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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

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MAXWELL, in his Electricity and Magnetism, gives diagrams showing the equipotential lines and lines of force surrounding certain two dimensional distributions of electrostatic charge. He remarks on the value of these diagrams in enabling us to form a rough idea of what will happen when we have a charged conductor of approximately the same shape as any of the equipotential lines of the diagram. In this paper, diagrams are presented of a few simple cases not given by Maxwell. Before giving a description of the diagrams themselves, a short account will be given of the method by which they were drawn.

It is well known that the potential due to a two dimensional distribution of electricity is continuous, and in empty space satisfies Laplace's equation,

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Within the body of the charge $\nabla^2 \phi = 4 \pi \rho$, and at surface charges the normal derivative jumps by $4 \eta \sigma$. Conversely, if we are given a continuous ϕ, then a distribution is uniquely determined satisfying the relations above. Either the charge determines the potential (except for an additive constant) or the potential determines the distribution.

In 1861 Newmann discussed the equation $\nabla^2 \phi = 0$. It appears that this is the necessary and sufficient condition that ϕ have a conjugate function ψ. This new function ψ also satisfies Laplace's equation, and its level lines are orthogonal to the lines of constant ϕ; that is, the lines of constant ψ are the lines of force of the distribution of which ϕ is the potential. It is evident then that if we are given two continuous conjugate functions, we may regard the level curves of either one of

them as the equipotential lines, and those of the other as the lines of force of a distribution determined by the equations above. In general, the distribution will depend upon which function is taken as the potential function.

It is a familiar fact that conjugate functions are additive; that is, if $\phi_1\ \psi_1$ and $\phi_2\ \psi_2$ are two sets of conjugates, then $\phi_1 + \phi_2$ is conjugate to $\psi_1 + \psi_2$. But the potential function is also additive. Hence if we know the potential and the force functions of two distributions separately, we may find the potential and force functions of the combination

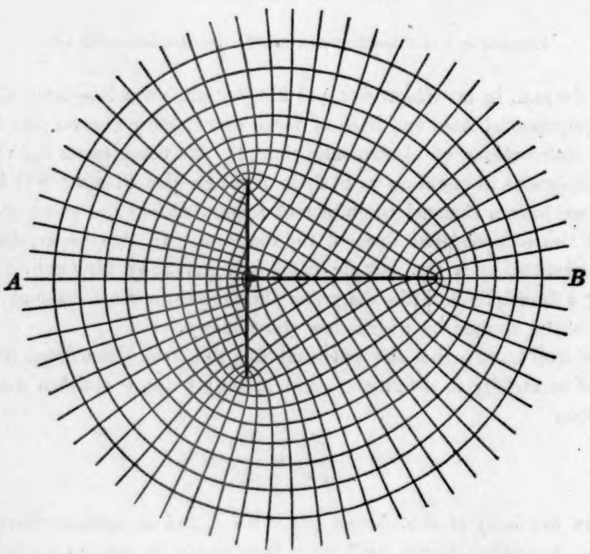

FIGURE 1.

by adding the functions of the distributions separately. This addition may be performed very simply by a graphical method described by Maxwell, provided that we know the shape of the curves of constant ϕ_1 and ϕ_2 separately. We draw the set of curves $\phi_1 = C_1$ for a set of values of C_1 differing by some constant, and the corresponding set $\phi_2 = C_2$, for successive values of C_2 differing by the same constant. We have thus divided the plane into a number of small curvilinear parallelograms, at every vertex of which we know the new potential $\phi_1 + \phi_2$. An equipotential line of $\phi_1 + \phi_2$ is obtained by drawing a

smooth curve through vertices at which $\phi_1 + \phi_2$ has the same value. This line will have the general direction of one of the diagonals in each parallelogram. The line of constant $\phi_1 - \phi_2$ is obtained from the same figure by following along the other diagonal. By an exactly similar method we find the lines of force $\psi_1 + \psi_2$ of the new combination. This is the method by which the curves of this paper were drawn. The diagrams represent the field surrounding two elements, the field of each of which singly is known.

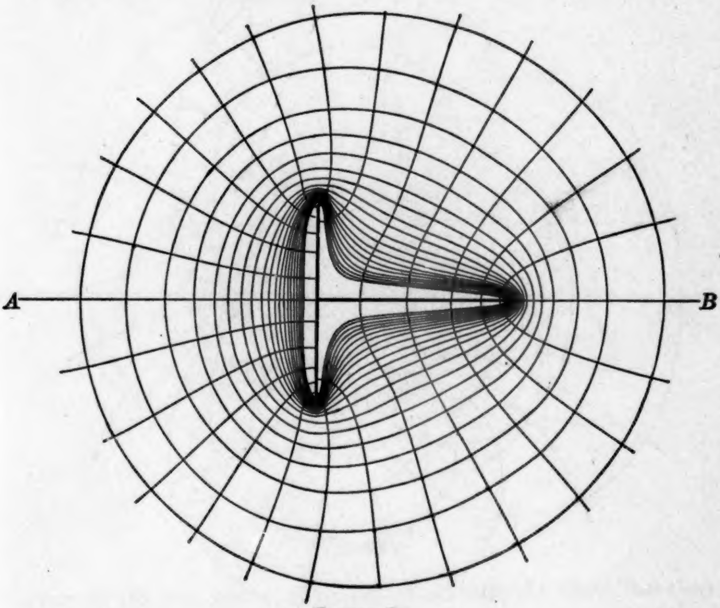

FIGURE 2.

The simple functions that were combined are:

$$\left.\begin{aligned} x &= \cosh\phi\cos\psi \\ y &= \sinh\phi\sin\psi \end{aligned}\right\} \quad \text{I}$$

x and y are conjugate functions of ϕ and ψ, and hence conversely ϕ and ψ are conjugate functions of x and y. Therefore both ϕ and ψ satisfy Laplace's equation, and if they are continuous, may be taken as poten-

tial functions. Eliminating ψ and ϕ successively between the equations above gives:

$$\left.\begin{aligned}\frac{x^2}{\cosh^2\phi}+\frac{y^2}{\sinh^2\phi}&=1\\ \frac{x^2}{\cos^2\psi}-\frac{y^2}{\sin^2\psi}&=1\end{aligned}\right\}$$

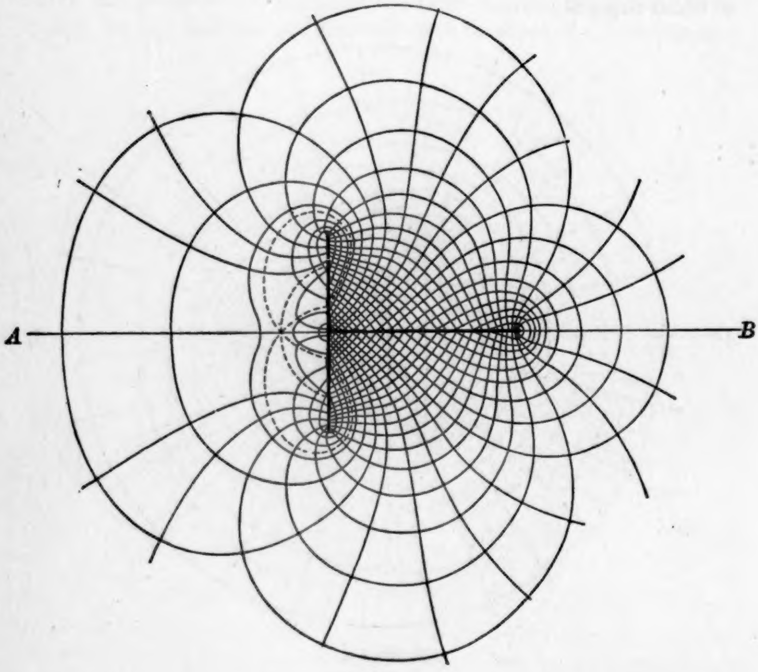

FIGURE 3.

ϕ defines a family of confocal ellipses, one of which reduces to the straight line joining (1, 0) with (−1, 0). ψ defines a family of hyperbolas with the same foci as the family of ellipses. One of the hyperbolas is the X axis with the segment joining (1, 0) to (−1, 0) omitted. ϕ and ψ define then a system of orthogonal curvilinear co-ordinates. Now if we examine the original equations I, we shall see that x and y do not uniquely define ϕ and ψ. In the first quadrant, for example, ϕ and ψ may be either both positive or both negative. This ambiguity enables

us so to assign values that either ϕ or ψ shall be continuous, and hence a potential function.

If ϕ is the potential, we give it the same value at every point on one of the ellipses. ϕ varies from 0 on the straight line joining the foci to infinity at infinity. The first derivatives are continuous everywhere

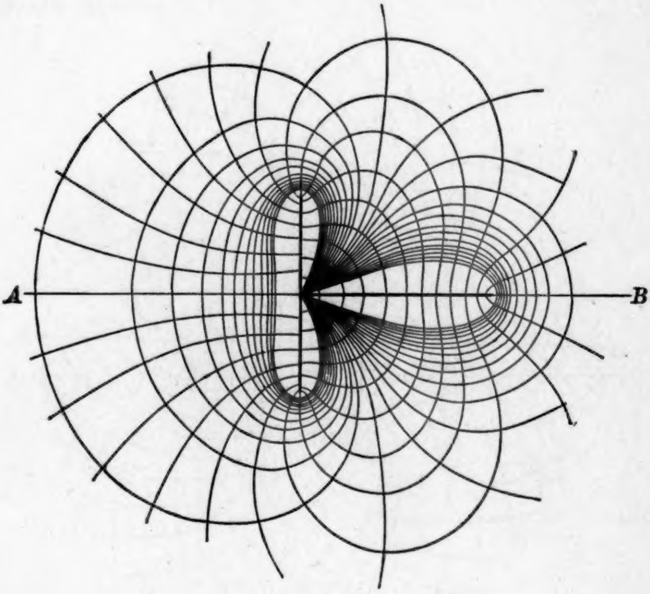

FIGURE 4.

except on the line, where the amount of discontinuity shows that there is a surface charge of

$$\sigma = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-x^2}},$$

counting the charge on both sides of the line. This gives a total charge all along the line of $\frac{1}{2}$. The values of ψ are now determined. Everywhere on the same branch of a hyperbola it has the same numerical value, but suffers an abrupt change of sign on passing through the vertex, being positive above and negative below the X axis. On the two different branches of the same hyperbola ψ has supplementary values.

We have therefore found the potential and lines of force due to a conducting board infinitely long, two units wide, and charged with half a unit of electricity per unit of length. Because of the shape of the equipotential lines, we may call this the elliptic element.

Now to make ψ the potential, it must be made continuous. This is done very simply by keeping the same distribution of values as before

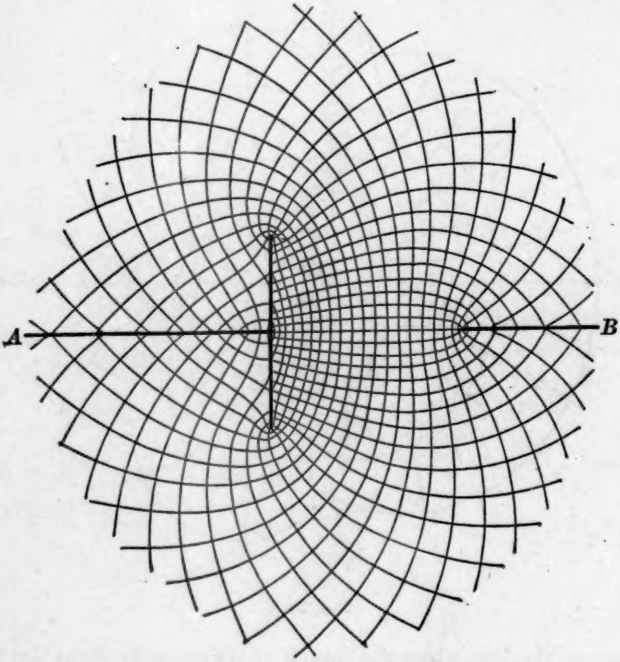

FIGURE 5.

in the first and second quadrants, while changing the sign in the third and fourth. ψ now has the same value all along one branch of a hyperbola and the supplementary value on the other branch. It increases in value from zero on the positive piece of the X axis to π on the negative part. The discontinuity shows that there is a surface charge on the interrupted X axis of

$$\frac{\pm 1}{2\pi\sqrt{x^2-1}},$$

being positive to the right, negative to the left. ϕ has now become discontinuous, it has the negative of its old values below the X axis. We have now found the potential and force functions due to an infinite conducting plane, with a strip two units wide removed, one half charged positively and the other negatively to the same density. This may be called the hyperbolic element, since the equipotential surfaces are hyperbolic cylinders.

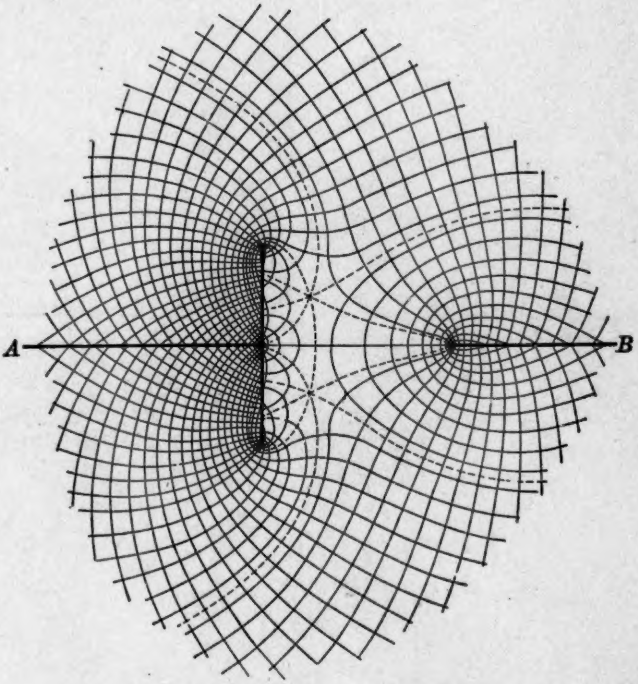

FIGURE 6.

The figures illustrate all possible combinations of two elliptic or hyperbolic elements, the centres being unit distance apart and the direction of one perpendicular to that of the other.

Figure 1 shows two elliptic elements charged with electricity of the same sign. It is to be noticed that the elements are no longer equipotential; that is, they can no longer be regarded as conducting. We must think of the charges as rigidly attached to the boards in the position

that they would naturally assume when free from external forces. It should be noted how, near the point of contact, one element dominates the other, a line of force springing across from one to the other, although the charges are of the same sign. This is because the surface density is infinite at the free end of an element.

Figure 2 is not a combination of any of the elements already described. The heavy lines represent rods charged uniformly with electricity of the

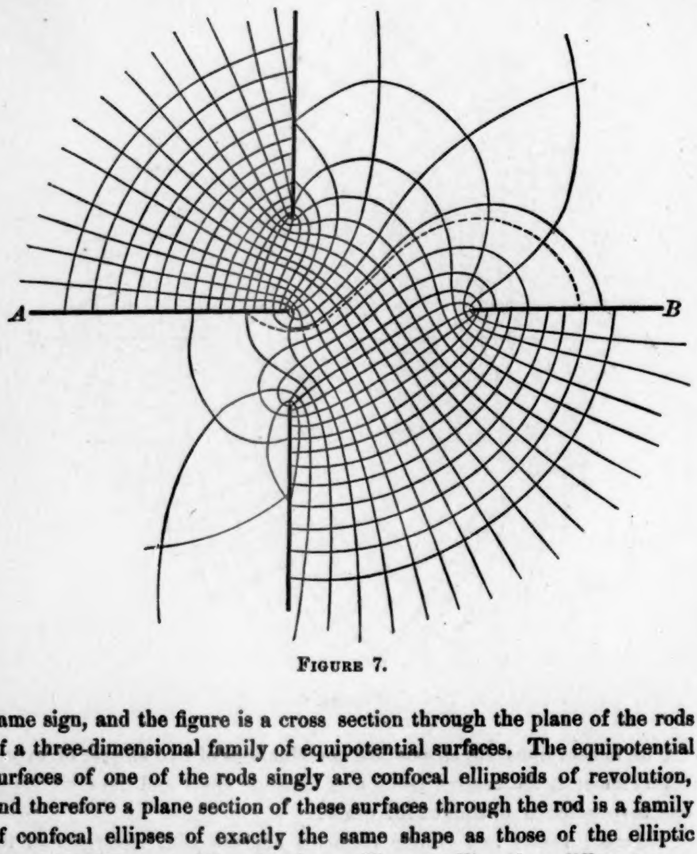

FIGURE 7.

same sign, and the figure is a cross section through the plane of the rods of a three-dimensional family of equipotential surfaces. The equipotential surfaces of one of the rods singly are confocal ellipsoids of revolution, and therefore a plane section of these surfaces through the rod is a family of confocal ellipses of exactly the same shape as those of the elliptic element. The potential varies from ellipse to ellipse in a different manner, however; for the elliptic element it remains finite, while it becomes infinite on approaching the rod. The figure was drawn to show how

changing the parameters of two families of curves, while keeping their shape constant, may change the shape of the family obtained by adding the parameters. At distances remote from the charges, the shape of the curves of Figures 1 and 2 are the same, but on approaching the rods the equipotential surfaces become infinitely close together, but never cut the rods. It was impossible to draw this, and the space about the rods was left vacant. The lines of force were drawn in free-hand. In three

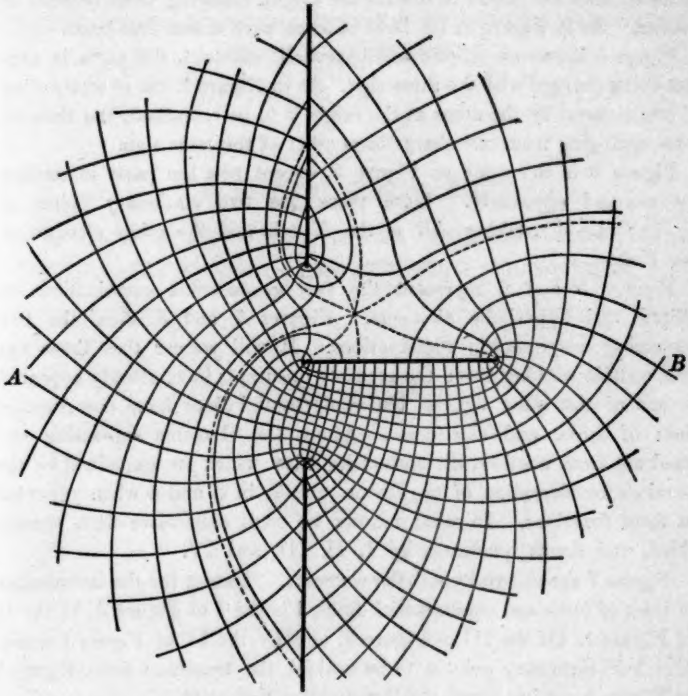

FIGURE 8.

dimensions there is no general additive force function corresponding to that in two dimensions.

Figure 3 represents the same combination of elements as Figure 1, except that now the signs of the charges are opposite. At the point, distant half a unit from the origin, through which the dotted lines pass, the potential has a stationary point. The lines of force point radially at this point from four perpendicular directions; at the point itself, the

magnitude of the force has sunk to zero. On the X axis, to the right of this point, the force is in the direction BA; to the left it has the direction AB.

Figure 4 is the same as Figure 2, except that the charges are of opposite signs. The line of zero potential has the same shape in both Figures 3 and 4. A curious survival of the tendency to the stationary point is seen as we come up toward the origin from A. The equipotential lines are drawn in toward the origin, changing from concave to convex. As in Figure 2, the lines of force were drawn free-hand.

Figure 5 shows an elliptic and hyperbolic element, the parts in contact being charged with the same sign. As in Figure 1, the overpowering of one element by the other at the origin is to be remarked, the lines of force springing from one charge to another of the same sign.

Figure 6 is the same as Figure 5, except now the parts in contact are charged oppositely. Here there are two stationary points at $(\frac{1}{2}, \frac{1}{2}) \cdot (\frac{1}{2}, -\frac{1}{2})$, taking AB as the X axis and the other element as the Y axis.

Figures 5 and 6 represent the two symmetrical combinations of elliptic and hyperbolic elements. Figures 7 and 8 show the two remaining unsymmetric combinations. It will appear that these figures contain nothing new; the separate quadrants have already appeared in one of the other figures, but equipotential lines have now become lines of force, and *vice versa*. Along the elements separating one quadrant from another, discontinuities arise, which are explained by the previous consideration of the discontinuities in ϕ and ψ when regarded as force functions. In what follows we shall abbreviate first, second, third, and fourth quadrants by I, II, III, and IV.

Figure 7 shows two hyperbolic elements. Except for the interchange of lines of force and equipotential lines, I is the I of Figure 3, II the II of Figure 1, III the III of Figure 3, and IV the IV of Figure 1 again. The half stationary point is to be noticed, the transition from Figure 1 to Figure 3 coming across the line dividing this point.

Figure 8 shows a hyperbolic and elliptic element. I is the I of Figure 6, II of Figure 5, III of Figure 6, IV of Figure 5. The potential here has only one stationary point, the halving of the number in Figure 6 corresponding to the halving of the single point of Figure 3.